

On the extension of Fredholm determinants to the mixed multidimensional integral operators with regulated kernels

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Abstract

We extend the classical trace (and determinant) known for the integral operators

$$(\mathcal{I}+) \int_{[0,1)^N} \mathbf{A}(\mathbf{k}, \mathbf{x}) \mathbf{u}(\mathbf{x}) d\mathbf{x}$$

with matrix-valued kernels \mathbf{A} to the operators of the form

$$\sum_{\alpha} \int_{[0,1)^{|\alpha|}} \mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha}) \mathbf{u}(\mathbf{k}_{\bar{\alpha}}, \mathbf{x}_{\alpha}) d\mathbf{x}_{\alpha},$$

where α are arbitrary subsets of the set $\{1, \dots, N\}$. Such operators form a Banach algebra containing simultaneously all integral operators of the dimensions $\leq N$. In this sense, it is a largest algebra where explicit traces and determinants are constructed. Such operators arise naturally in the mechanics and physics of waves propagating through periodic structures with various defects. We give an explicit representation of the inverse operators (resolvent) and describe the spectrum by using zeroes of the determinants. Due to the structure of the operators, we have 2^N different determinants, each of them describes the spectral component of the corresponding dimension.

Keywords: operator algebras, traces and determinants, periodic lattice with defects, guided and localized waves

1. Introduction

Defects in periodic structures and corresponding periodic operators play a significant role in various fields of science, see, e.g. [1, 2, 3, 4, 5]. It is shown in [6, 7] that periodic operators with crossing defects (periodic sublattices) of various dimensions are unitarily equivalent to the integral operators of the special form. We start with introducing these integral operators. At first, we consider the operators with staircase (piecewise constant) kernels because such kernels significantly simplify the proof of key Lemma 2.6 and they are important in numerical applications. Finally, we generalize our results to the continuous (or even regulated) kernels. Let N, M be some positive integer numbers. Introduce the

Hilbert space $L_{N,M}^2$ of square-integrable vector-valued (if $M > 1$) functions acting on the cube $[0, 1]^N$ and integrals $\langle \dots \rangle_\alpha$ by

$$L_{N,M}^2 := L^2([0, 1]^N, \mathbb{C}^M), \quad \langle \dots \rangle_\alpha := \int_{[0,1]^{|\alpha|}} \dots d\mathbf{x}_\alpha \quad (1)$$

where $\alpha \in I_N = 2^{\{1, \dots, N\}}$ is a subset of the set $\{1, \dots, N\}$, $|\alpha|$ is the number of elements of α , and $d\mathbf{x}_\alpha = \prod_{i \in \alpha} dx_i$, and $\langle \mathbf{a} \rangle_\emptyset := \mathbf{a}$ for any \mathbf{a} . The dots \dots in (1) means any matrix-valued function depending on $\mathbf{x}_\alpha = (x_i)_{i \in \alpha} \in [0, 1]^{|\alpha|}$, the components x_i in \mathbf{x}_α are arranged in the order of increasing indices. Let α, β be disjoint subsets from I_N . It is convenient to use the following notation

$$\mathbf{y}_\beta \diamond \mathbf{x}_\alpha = \mathbf{x}_\alpha \diamond \mathbf{y}_\beta = \mathbf{z}_{\alpha \cup \beta}, \quad \text{where } z_i = \begin{cases} x_i, & i \in \alpha, \\ y_i, & i \in \beta. \end{cases} \quad (2)$$

All components of $\mathbf{x}_\alpha, \mathbf{y}_\beta, \mathbf{z}_{\alpha \cup \beta}$ in (2) are arranged in the order of increasing indices. The operation \diamond is also associative. Define the following subset of the algebra $\mathcal{B}_{N,M}$ of all bounded operators acting on $L_{N,M}^2$. Let $h = 1/p$ with some $p \in \mathbb{N}$. Define $\chi_i^h(y)$ is a function which is 1 for $y \in [i/p, (i+1)/p)$ and is 0 otherwise. We call the function $A(y_1, \dots, y_R)$ as h -staircase if it is a linear combination of $\prod_{j=1}^R \chi_{i_j}^h(y_j)$. We call the matrix-valued function $\mathbf{A}(y_1, \dots, y_M)$ as h -staircase if all entries are h -staircase.

Definition 1.1. Let $\mathcal{L}_{N,M}^h$ be the set

$$\mathcal{L}_{N,M}^h = \{ \mathcal{A} : \mathcal{A} = \sum_{\alpha \in I_N} \langle \mathbf{A}_\alpha(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha \}, \quad (3)$$

where \mathbf{A}_α are any h -staircase $M \times M$ matrix-valued functions depending on $\mathbf{k} = (k_i)_{i=1}^N \in [0, 1]^N$, $\mathbf{x}_\alpha = (x_i)_{i \in \alpha} \in [0, 1]^{|\alpha|}$. The dot \cdot denotes the place of the operator argument, i.e.

$$\mathcal{A}\mathbf{u}(\mathbf{k}) = \sum_{\alpha \in I_N} \langle \mathbf{A}_\alpha(\mathbf{k}, \mathbf{x}_\alpha) \mathbf{u}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_\alpha) \rangle_\alpha, \quad \mathbf{u} \in L_{N,M}^2, \quad (4)$$

where $\bar{\alpha}$ is the complement to the set $\alpha \in I_N$.

It is obvious that \mathcal{L} (sometimes we will omit indices N, M and h for convenience) is a linear subspace of \mathcal{B} . Let us introduce the following positive function on \mathcal{L} .

Definition 1.2. Denote

$$\|\mathcal{A}\|_{\mathcal{L}} = \sum_{\alpha} \max_{(\mathbf{k}, \mathbf{x}_\alpha), i} \sum_{j=1}^M |a_{i,j,\alpha}(\mathbf{k}, \mathbf{x}_\alpha)|, \quad (5)$$

where $\mathbf{A}_\alpha = (a_{i,j,\alpha})_{i,j=1}^M$. Due to Lemma 2.1, definition (5) is correct.

The next proposition describes the basic properties of \mathcal{L} .

Theorem 1.3. *The function $\|\dots\|_{\mathcal{L}}$ is a norm on \mathcal{L} . This norm is stronger than the standard operator norm $\|\dots\|_{\mathcal{B}}$. The convergence in $\|\dots\|_{\mathcal{L}}$ is equivalent to the uniform convergence of the coefficients \mathbf{A}_{α} . The structure $(\mathcal{L}, \|\dots\|_{\mathcal{L}})$ is a Banach algebra (associative, non-commutative).*

While the sum of two operators from \mathcal{L} leads to the sum of the corresponding matrix-valued functions in their representations, the product (composition) is more complicated. For example, it is not difficult to check the following identity for the product of two terms

$$\langle \mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha}) \cdot \rangle_{\alpha} \langle \mathbf{B}(\mathbf{k}, \mathbf{x}_{\beta}) \cdot \rangle_{\beta} = \langle (\mathbf{A} \circ \mathbf{B})(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta}) \cdot \rangle_{\alpha \cup \beta}, \quad (6)$$

where the matrix-valued function $\mathbf{A} \circ \mathbf{B}$ is defined by

$$(\mathbf{A} \circ \mathbf{B})(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta}) = \int_{[0,1]^{|\alpha \cap \beta|}} \mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha \setminus \beta} \diamond \mathbf{z}_{\alpha \cap \beta}) \mathbf{B}(\mathbf{k}_{\alpha} \diamond \mathbf{z}_{\alpha \cap \beta} \diamond \mathbf{x}_{\alpha \setminus \beta}, \mathbf{x}_{\beta}) d\mathbf{z}_{\alpha \cap \beta}. \quad (7)$$

Now, define the "building blocks" of \mathcal{L} and $\text{Inv}(\mathcal{L})$ (invertible operators). We assume the invertibility in \mathcal{B} but we show below that if $\mathcal{A} \in \mathcal{L}$ is invertible in \mathcal{B} then $\mathcal{A}^{-1} \in \mathcal{L}$. This is more or less evident for finite-dimensional \mathcal{L} . Introduce

$$\mathcal{L}_{\alpha} = \{ \langle \mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha}) \cdot \rangle_{\alpha} : \forall \mathbf{A} \} \subset \mathcal{L}; \quad \mathcal{G}_0 = \text{Inv}(\mathcal{L}_0), \quad \mathcal{G}_{\alpha} = \text{Inv}(\mathcal{I} + \mathcal{L}_{\alpha}) \text{ if } \alpha \neq \emptyset, \quad (8)$$

where $\mathcal{I} := \mathbf{I} \cdot$ is the identity operator and \mathbf{I} is the identity matrix.

Theorem 1.4. *The sets \mathcal{L}_{α} are closed subalgebras of \mathcal{L} , the sets \mathcal{G}_{α} are closed subgroups of $\text{Inv}(\mathcal{L})$, satisfying*

$$\mathcal{L}_{\alpha} \cap \mathcal{L}_{\beta} = \{0\}, \quad \mathcal{G}_{\alpha} \cap \mathcal{G}_{\beta} = \{\mathcal{I}\} \quad \text{for } \alpha \neq \beta. \quad (9)$$

The following identities hold true

$$\mathcal{L}_{\alpha} \mathcal{L}_{\beta} \subset \mathcal{L}_{\alpha \cup \beta}, \quad \mathcal{L} = \sum_{\alpha \in I_N} \mathcal{L}_{\alpha}, \quad \text{Inv}(\mathcal{L}) = \prod_{\alpha \in I_N} \mathcal{G}_{\alpha}, \quad (10)$$

where the order of terms in the sum is not important but the terms in the product are arranged in ascending (or descending) order of $|\alpha|$. Moreover, for given \mathcal{A} and for given order of terms the corresponding representations as the sum and the product of elementary operators are unique.

The explicit procedure of finding the components in the product (10) is given in the proof of Theorem 1.4. Because the inverse of the elementary operators from the product have an explicit form, we obtain the explicit form for the inverse of the product of operators. This means that having \mathcal{A} we can check its invertibility and find \mathcal{A}^{-1} explicitly. Before introducing

the trace and the determinant, let us introduce the commutative algebras of complex scalar h -staircase functions acting on different cubes $[0, 1]^r$

$$\mathcal{C}_{\alpha, N}^h = \{f(\mathbf{k}_{\bar{\alpha}}), \mathbf{k}_{\bar{\alpha}} = (k_i)_{i \notin \alpha} \in [0, 1]^{N-|\alpha|}\}, \quad \mathcal{C}_N^h = \bigoplus_{\alpha \in I_N} \mathcal{C}_{\alpha, N}^h. \quad (11)$$

The algebra $\mathcal{C}_{\emptyset, N}$ is the algebra of all complex h -staircase scalar functions defined on the cube $[0, 1]^N$. If $\alpha = \{1, \dots, N\}$ then we put $\mathcal{C}_{\alpha, N} := \mathbb{C}$. All $\mathcal{C}_{\alpha, N}$ are subalgebras of $\mathcal{C}_{\emptyset, N}$ and they consist of the functions independent on some variables k_i . In this sense if $|\alpha| = |\beta|$ but $\alpha \neq \beta$ then $\mathcal{C}_{\alpha} \neq \mathcal{C}_{\beta}$. All operations $+$, $-$, $*$, $\bar{\cdot}$ (complex conjugation), $\exp(\cdot)$, etc in the algebra \mathcal{C} are assumed to be componentwise. Taking the standard norm $\|\mathbf{f}\| = \max_{\alpha, \mathbf{k}_{\bar{\alpha}}} |f_{\alpha}(\mathbf{k}_{\bar{\alpha}})|$ we construct the commutative finite-dimensional Banach algebra $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$. Introduce the following mappings.

Definition 1.5. *Define the following mapping*

$$\tau : \mathcal{L} \rightarrow \mathcal{C}, \quad \tau(\mathcal{A}) = (\tau_{\alpha}(\mathcal{A})), \quad \tau_{\alpha}(\mathcal{A}) = \tau_{\alpha}(\mathcal{A})(\mathbf{k}_{\bar{\alpha}}) := \text{Tr} \langle \mathbf{A}_{\alpha}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_{\alpha}, \mathbf{x}_{\alpha}) \rangle_{\alpha}. \quad (12)$$

Now, we fix some ascending (or descending) order of $|\alpha|$. Below we show that the definition does not depend on the order. If $\mathcal{A} \in \text{Inv}(\mathcal{L})$ then by Theorem 1.4 $\mathcal{A} = \prod_{\alpha \in I_N} \mathcal{G}_{\alpha}$, where $\mathcal{G}_{\alpha} = \mathcal{I} + \langle \mathbf{G}_{\alpha}(\mathbf{k}, \mathbf{x}_{\alpha}) \cdot \rangle_{\alpha}$. We can write $\mathcal{G}_{\alpha} = \mathcal{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}})$, where for any fixed $\mathbf{k}_{\bar{\alpha}}$ the operator $\mathcal{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}}) = \mathcal{I} + \langle \mathbf{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{k}_{\alpha}, \mathbf{x}_{\alpha}) \cdot \rangle_{\alpha}$ (or $\mathcal{G}_{\emptyset}(\mathbf{k}) = \mathbf{G}_{\emptyset}(\mathbf{k}) \cdot$ for $\alpha = \emptyset$) is a finite-rank operator acting on $L^2([0, 1]^{|\alpha|}, \mathbb{C}^M)$. Hence, we can define the determinant π_{α} of such operators

$$\pi_{\alpha}(\mathcal{G}_{\alpha}) = \pi_{\alpha}(\mathcal{G}_{\alpha})(\mathbf{k}_{\bar{\alpha}}) := \pi_{\alpha}(\mathcal{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}})) \quad (13)$$

This leads to the mapping

$$\pi : \text{Inv}(\mathcal{L}) \rightarrow \text{Inv}(\mathcal{C}), \quad \pi(\mathcal{A}) = (\pi_{\alpha}(\mathcal{G}_{\alpha})), \quad (14)$$

since $\mathcal{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}})$, and hence their determinants, are staircase functions. The set $\text{Inv}(\mathcal{C})$ consists of all functions from \mathcal{C} that have no zeroes, since the determinant of finite-rank invertible operators is not zero.

Remark (computation of π). While τ is simple, the computation of π is more complicated. Nevertheless, there are various formulas for π . Separating the variables (see the proof of Lemma 2.2) $\mathcal{I} + \langle \mathbf{G}(\mathbf{k}, \mathbf{x}_{\alpha}) \cdot \rangle_{\alpha} = \mathcal{I} + \mathbf{C}(\mathbf{k}) \langle \mathbf{D}(\mathbf{x}_{\alpha}) \cdot \rangle_{\alpha}$ we can compute the determinant

$$\pi_{\alpha}(\mathcal{I} + \langle \mathbf{G}(\mathbf{k}, \mathbf{x}_{\alpha}) \cdot \rangle_{\alpha}) = \det \mathbf{E}(\mathbf{k}_{\bar{\alpha}}), \quad \mathbf{E}(\mathbf{k}_{\bar{\alpha}}) = \mathbf{I} + \langle \mathbf{D}(\mathbf{x}_{\alpha}) \mathbf{C}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_{\alpha}) \rangle_{\alpha}. \quad (15)$$

This is a very convenient computation of the determinant of finite-rank operators, see, e.g. [8], [9]. We can use the Fredholm formulas

$$\pi_{\alpha}(\mathcal{I} + \langle \mathbf{G}_{\alpha}(\mathbf{k}, \mathbf{x}_{\alpha}) \cdot \rangle_{\alpha}) = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{[0, 1]^{n|\alpha|}} P_n(\mathbf{k}_{\bar{\alpha}}, \mathbf{x}_{1\alpha}, \dots, \mathbf{x}_{n\alpha}) d\mathbf{x}_{1\alpha} \dots d\mathbf{x}_{n\alpha}, \quad (16)$$

$$P_n(\mathbf{k}_{\bar{\alpha}}, \mathbf{x}_{1\alpha}, \dots, \mathbf{x}_{n\alpha}) = \det \begin{pmatrix} \mathbf{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_{1\alpha}, \mathbf{x}_{1\alpha}) & \dots & \mathbf{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_{1\alpha}, \mathbf{x}_{n\alpha}) \\ \vdots & \dots & \vdots \\ \mathbf{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_{n\alpha}, \mathbf{x}_{1\alpha}) & \dots & \mathbf{G}_{\alpha}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_{n\alpha}, \mathbf{x}_{n\alpha}) \end{pmatrix} \quad (17)$$

which also lead to the determinant, see, e.g., [10, 8].

The next theorem shows us that τ and π are the trace and the determinant (totally, i.e. not only for the elementary operators from \mathcal{G}_{α}). The theory of traces and determinants has own special interest, see, e.g. [8, 11, 12, 13, 14]. In our case, we extend the traces and determinants to the algebra \mathcal{L} containing simultaneously operators of multiplication by matrix-valued functions and various classes of integral operators. That is why, in our case, the trace and the determinant are vectors consisting of some functions (usually, traces and determinants are just numbers). At the end of this section we discuss how to extend the trace and the determinant to the operators with "general" kernels.

Theorem 1.6. *For any $\lambda, \mu \in \mathbb{C}$, $\mathcal{A}, \mathcal{B} \in \mathcal{L}$, and $\mathcal{C}, \mathcal{D} \in \text{Inv}(\mathcal{L})$ the following identities hold true*

$$\tau(\lambda\mathcal{A} + \mu\mathcal{B}) = \lambda\tau(\mathcal{A}) + \mu\tau(\mathcal{B}), \quad \tau(\mathcal{A}\mathcal{B}) = \tau(\mathcal{B}\mathcal{A}), \quad \pi(\mathcal{C}\mathcal{D}) = \pi(\mathcal{C})\pi(\mathcal{D}). \quad (18)$$

Moreover τ, π are continuous mappings satisfying $\pi(e^{\mathcal{A}}) = e^{\tau(\mathcal{A})}$ and $\|\tau\|_{\mathcal{L} \rightarrow \mathcal{E}} = M$.

Results of Theorem 1.6 mean that τ and π are the trace and the determinant. They are in a good agreement with the standard trace and determinant of finite-rank operators. In this sense they are unique. Of course, taking linear combinations of τ_{α} and products of π_{α} we can construct other traces and determinants but only π and τ contain the most complete information about the operator. Since $\pi(e^{\mathcal{A}}) = e^{\tau(\mathcal{A})}$ we can use the analog of Plemelj-Smithies formula

$$\pi_{\alpha}(\mathcal{I} + \mathcal{A}) = 1 + \sum_{n=1}^{+\infty} \frac{\det}{n!} \begin{pmatrix} \tau_{\alpha}(\mathcal{A}) & n-1 & 0 & 0 & \dots & 0 & 0 \\ \tau_{\alpha}(\mathcal{A}^2) & \tau_{\alpha}(\mathcal{A}) & n-2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \tau_{\alpha}(\mathcal{A}^{n-1}) & \tau_{\alpha}(\mathcal{A}^{n-2}) & \tau_{\alpha}(\mathcal{A}^{n-3}) & \dots & \dots & \tau_{\alpha}(\mathcal{A}) & 1 \\ \tau_{\alpha}(\mathcal{A}^n) & \tau_{\alpha}(\mathcal{A}^{n-1}) & \tau_{\alpha}(\mathcal{A}^{n-2}) & \dots & \dots & \tau_{\alpha}(\mathcal{A}^2) & \tau_{\alpha}(\mathcal{A}) \end{pmatrix}. \quad (19)$$

It is seen that the spectrum

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \exists (\lambda\mathcal{I} - \mathcal{A})^{-1}\} \quad (20)$$

can be determined as zeroes of the continuously extended determinant

$$\sigma(\mathcal{A}) = \bigcup_{\alpha \in I_N} \{\lambda \in \mathbb{C} : \pi_{\alpha}(\lambda\mathcal{I} - \mathcal{A})(\mathbf{k}_{\bar{\alpha}}) = 0 \text{ for some } \mathbf{k}_{\bar{\alpha}}\} \quad (21)$$

because if all $\pi_{\alpha}(\mathcal{A})$ are non-zero for some \mathcal{A} then \mathcal{A} is invertible (see (41) and below). The identities $\pi_{\alpha}(\lambda\mathcal{I} - \mathcal{A}) = 0$ lead to implicit functions $\lambda = \lambda_{\bar{\alpha}}(\mathbf{k}_{\bar{\alpha}})$ which are very important in

the study of mechanics and physics of waves (about spectral problems in wave dynamics see, e.g., [15, 16, 17]). The functions $\lambda_{\bar{\alpha}}$ generalize the well-known Floquet-Bloch (F-B) dispersion branches and describe the dependence of the spectral parameter λ (energy, frequency) on the wave number (or quasimomentum) $\mathbf{k}_{\bar{\alpha}}$ corresponding to the defect modes. Defect modes are non-attenuated (quasiperiodic $\sim e^{i\mathbf{k}_{\bar{\alpha}}\mathbf{n}_{\bar{\alpha}}}$, where $\mathbf{n}_{\bar{\alpha}} \in \mathbb{Z}^{|\bar{\alpha}|}$ parameterize the defect sublattice) along the defect and exponentially decaying in the perpendicular directions (so-called guided or Rayleigh modes). Due to the attenuation, the dispersion branches $\lambda_{\bar{\alpha}}$ do not depend on some components of the full quasimomentum \mathbf{k} . This is a significant difference between the periodic structures with defects of lower dimensions and purely periodic structures without defects, where there are no attenuation of the modes and F-B dispersion branches depend on the whole \mathbf{k} , i.e. we have only $\lambda_{\emptyset}(\mathbf{k})$.

Important remark. Note that

$$\mathcal{L}_{N,M}^{\frac{1}{n}} \cup \mathcal{L}_{N,M}^{\frac{1}{m}} \subset \mathcal{L}_{N,M}^{\frac{1}{nm}}, \quad n, m \in \mathbb{N}. \quad (22)$$

So we can consider the algebra

$$\mathcal{L}_{N,M}^0 = \bigcup_{n=1}^{\infty} \mathcal{L}_{N,M}^{\frac{1}{n}}. \quad (23)$$

Taking the completeness of \mathcal{L}^0 by the norm $\|\cdot\|_{\mathcal{L}}$ (which is the same for all $\mathcal{L}_{N,M}^{\frac{1}{n}}$) we obtain the Banach algebra $\overline{\mathcal{L}}_{N,M}^0$ of multidimensional integral operators with regulated kernels (possible discontinuities are in rational points). In particular, this algebra contains the operators with continuous kernels. The same procedure leads to $\overline{\mathcal{C}}^0$ consisting of regulated vector-valued functions. It is seen that the trace τ is continuous in $\overline{\mathcal{L}}_{N,M}^0$. Hence, we can continuously extend the determinant π to the operators from $\text{Inv}(\overline{\mathcal{L}}_{N,M}^0)$. This reminds the Schmidt idea (see, e.g., [18], 2.4 The Fredholm Theorems, p. 48) of approximation of the integral kernel by a separable kernel and a small kernel which leads to two integral equations with explicit solutions. Note that taking the other systems of intervals (not only $[i/p, i/p + 1/p)$) we can obtain different classes of regulated kernels for which the traces and determinants are defined.

2. Proof of the results

Lemma 2.1. *If $\mathcal{A} = 0$ then all $\mathbf{A}_{\alpha} = 0$, i.e. the representation (2) is unique.*

Proof. The proof repeats the arguments from [6, 7, 9]. We give it in a short form. The proof consists of 2^N steps. On the first step we suppose that $\mathbf{A}_{\emptyset} \neq 0$. Then there is $\mathbf{k}^0 = (k_1^0, \dots, k_N^0) \in (0, 1)^N$ such that $\mathbf{A}_0(\mathbf{k}^0)\mathbf{f} \neq 0$ for some constant $\mathbf{f} \in \mathbb{C}^M$. Consider the functions

$$\eta_i(k_i) = \frac{1}{\sqrt{2\varepsilon}} \begin{cases} 1, & k_i \in [k_i^0 - \varepsilon, k_i^0 + \varepsilon], \\ 0, & \text{otherwise,} \end{cases} \quad (24)$$

where $\varepsilon > 0$ is some small number. Define $\eta(\mathbf{k}) = \prod_{i=1}^N \eta_i(k_i)$. It is not difficult to check that $\langle \mathbf{A}_\alpha \mathbf{f} \eta \rangle_\alpha$ (where $\alpha \neq \emptyset$) are small for small ε because integrals $\langle \eta \rangle_\alpha$ are small. Due to $\mathbf{A}_0(\mathbf{k}^0) \mathbf{f} \neq 0$ we have $\mathcal{A} \mathbf{f} \eta = \mathbf{A}_\emptyset \mathbf{f} \eta + \sum_{\alpha \in I_N \setminus \emptyset} \langle \mathbf{A}_\alpha \mathbf{f} \eta \rangle_\alpha$ is not small for all small ε . This is contrary to $\mathcal{A} = 0$. Then $\mathbf{A}_\emptyset = 0$. On the next steps we consistently obtain $\mathbf{A}_{\{i\}} = 0$, after that $\mathbf{A}_{\{i,j\}} = 0$ and so on. For this we only need to take $\eta_\alpha = \prod_{i \in \alpha} \eta_i$ instead of η and use the same arguments as above. ■

Proof of Theorem 1.3. For any matrix $\mathbf{A} = (a_{i,j})_{i,j=1}^M \in \mathbb{C}^{M \times M}$ denote

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^M |a_{i,j}|. \quad (25)$$

It is a norm satisfying $\|\mathbf{A}\mathbf{B}\|_\infty \leq \|\mathbf{A}\|_\infty \|\mathbf{B}\|_\infty$. The function (5) is

$$\|\mathcal{A}\|_{\mathcal{L}} = \sum_{\alpha \in I_N} \|\langle \mathbf{A}_\alpha(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha\|_{\mathcal{L}} = \sum_{\alpha \in I_N} \max_{(\mathbf{k}, \mathbf{x}_\alpha)} \|\mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha)\|_\infty. \quad (26)$$

It is obvious that $\|\lambda \mathcal{A}\|_{\mathcal{L}} = |\lambda| \|\mathcal{A}\|_{\mathcal{L}}$. Due to Lemma 2.1 we have $\|\mathcal{A}\|_{\mathcal{L}} = 0$ if and only if $\mathcal{A} = 0$. Consider two operators $\mathcal{A}, \mathcal{B} \in \mathcal{L}$

$$\mathcal{A} = \sum_{\alpha \in I_N} \langle \mathbf{A}_\alpha(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha, \quad \mathcal{B} = \sum_{\alpha \in I_N} \langle \mathbf{B}_\alpha(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha. \quad (27)$$

Then

$$\|\mathcal{A} + \mathcal{B}\|_{\mathcal{L}} = \sum_{\alpha} \max_{(\mathbf{k}, \mathbf{x}_\alpha)} \|(\mathbf{A} + \mathbf{B})(\mathbf{k}, \mathbf{x}_\alpha)\|_\infty \leq \sum_{\alpha} \max_{(\mathbf{k}, \mathbf{x}_\alpha)} (\|\mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha)\|_\infty + \|\mathbf{B}(\mathbf{k}, \mathbf{x}_\alpha)\|_\infty) \quad (28)$$

$$\leq \sum_{\alpha} (\max_{(\mathbf{k}, \mathbf{x}_\alpha)} \|\mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha)\|_\infty + \max_{(\mathbf{k}, \mathbf{x}_\alpha)} \|\mathbf{B}(\mathbf{k}, \mathbf{x}_\alpha)\|_\infty) = \|\mathcal{A}\|_{\mathcal{L}} + \|\mathcal{B}\|_{\mathcal{L}}. \quad (29)$$

Due to (6), (7), (26) we have

$$\|\langle \mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha \langle \mathbf{B}(\mathbf{k}, \mathbf{x}_\beta) \cdot \rangle_\beta\|_{\mathcal{L}} = \|\langle (\mathbf{A} \circ \mathbf{B})(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta}) \cdot \rangle_{\alpha \cup \beta}\|_{\mathcal{L}} \quad (30)$$

$$= \max_{(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta})} \left\| \int_{[0,1]^{|\alpha \cap \beta|}} \mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha \setminus \beta} \diamond \mathbf{z}_{\alpha \cap \beta}) \mathbf{B}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{z}_{\alpha \cap \beta} \diamond \mathbf{x}_{\alpha \setminus \beta}, \mathbf{x}_\beta) d\mathbf{z}_{\alpha \cap \beta} \right\|_\infty \quad (31)$$

$$\leq \max_{(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta})} \int_{[0,1]^{|\alpha \cap \beta|}} \|\mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha \setminus \beta} \diamond \mathbf{z}_{\alpha \cap \beta}) \mathbf{B}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{z}_{\alpha \cap \beta} \diamond \mathbf{x}_{\alpha \setminus \beta}, \mathbf{x}_\beta)\|_\infty d\mathbf{z}_{\alpha \cap \beta} \quad (32)$$

$$\leq \max_{(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta})} \max_{\mathbf{z}_{\alpha \cap \beta}} \|\mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha \setminus \beta} \diamond \mathbf{z}_{\alpha \cap \beta}) \mathbf{B}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{z}_{\alpha \cap \beta} \diamond \mathbf{x}_{\alpha \setminus \beta}, \mathbf{x}_\beta)\|_\infty \quad (33)$$

$$\leq \max_{(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta})} \max_{\mathbf{z}_{\alpha \cap \beta}} \|\mathbf{A}(\mathbf{k}, \mathbf{x}_{\alpha \setminus \beta} \diamond \mathbf{z}_{\alpha \cap \beta})\|_\infty \|\mathbf{B}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{z}_{\alpha \cap \beta} \diamond \mathbf{x}_{\alpha \setminus \beta}, \mathbf{x}_\beta)\|_\infty \quad (34)$$

$$\leq \max_{(\mathbf{k}, \mathbf{x}_\alpha)} \|\mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha)\|_\infty \max_{(\mathbf{k}, \mathbf{x}_\beta)} \|\mathbf{B}(\mathbf{k}, \mathbf{x}_\beta)\|_\infty = \|\langle \mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha\|_{\mathcal{L}} \|\langle \mathbf{B}(\mathbf{k}, \mathbf{x}_\beta) \cdot \rangle_\beta\|_{\mathcal{L}} \quad (35)$$

Due to (6) we have

$$\mathcal{AB} = \mathcal{C} = \sum_{\alpha \in I_N} \mathcal{C}_\alpha, \quad \mathcal{C}_\alpha = \sum_{\gamma, \delta: \gamma \cup \delta = \alpha} \langle \mathbf{A}_\gamma \cdot \rangle_\gamma \langle \mathbf{B}_\delta \cdot \rangle_\delta. \quad (36)$$

Using (26), (36), (30)-(35) we obtain

$$\|\mathcal{AB}\|_{\mathcal{L}} \leq \sum_{\alpha} \|\mathcal{C}_\alpha\|_{\mathcal{L}} \leq \sum_{\alpha} \sum_{\gamma, \delta: \gamma \cup \delta = \alpha} \|\langle \mathbf{A}_\gamma \cdot \rangle_\gamma \langle \mathbf{B}_\delta \cdot \rangle_\delta\|_{\mathcal{L}} \quad (37)$$

$$\leq \sum_{\alpha} \sum_{\gamma, \delta: \gamma \cup \delta = \alpha} \|\langle \mathbf{A}_\gamma \cdot \rangle_\gamma\|_{\mathcal{L}} \|\langle \mathbf{B}_\delta \cdot \rangle_\delta\|_{\mathcal{L}} = \left(\sum_{\alpha} \|\langle \mathbf{A}_\alpha \cdot \rangle_\alpha\|_{\mathcal{L}} \right) \left(\sum_{\alpha} \|\langle \mathbf{B}_\alpha \cdot \rangle_\alpha\|_{\mathcal{L}} \right) \quad (38)$$

$$= \|\mathcal{A}\|_{\mathcal{L}} \|\mathcal{B}\|_{\mathcal{L}}. \quad (39)$$

The identities (37)-(39) show that \mathcal{L} is a Banach algebra. ■

Lemma 2.2. *i) If $\mathcal{A} = \mathbf{A}_\emptyset(\mathbf{k}) \cdot \in \mathcal{H}_\emptyset$ is invertible then $\mathbf{A}_\emptyset(\mathbf{k})$ is invertible $\forall \mathbf{k}$ and $\mathcal{A}^{-1} = \mathbf{A}_\emptyset^{-1}(\mathbf{k}) \cdot \in \mathcal{H}_\emptyset$. ii) If $\mathcal{A} \in (\mathcal{I} + \mathcal{L}_\alpha)$ is invertible in \mathcal{B} then $\mathcal{A}^{-1} \in (\mathcal{I} + \mathcal{L}_\alpha)$ as well.*

Proof. The statement i) is simple, see, e.g. [6]. While the case of continuous kernels is considered in [6], the proof for the staircase kernels is the same. We just note that if \mathbf{A}_\emptyset is h -staircase then $\mathbf{A}_\emptyset^{-1}(\mathbf{k})$ is also h -staircase.

ii) It is true that $\mathcal{A} = \mathcal{I} + \langle \mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha$ for some \mathbf{A} . Because \mathbf{A} is h -staircase, we can separate variables $\mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha) = \mathbf{C}(\mathbf{k})\mathbf{D}(\mathbf{x}_\alpha)$ with some h -staircase \mathbf{C} , \mathbf{D} . Note that the matrices \mathbf{C} , \mathbf{D} can be non-square. For the operators with separable kernels there is an explicit representation for the inverse operators (see, e.g., [6])

$$\mathcal{A}^{-1} = (\mathcal{I} + \mathbf{C}(\mathbf{k}) \langle \mathbf{D}(\mathbf{x}_\alpha) \cdot \rangle_\alpha)^{-1} = \mathcal{I} - \mathbf{C}(\mathbf{k}) \mathbf{E}^{-1}(\mathbf{k}_{\bar{\alpha}}) \langle \mathbf{D}(\mathbf{x}_\alpha) \cdot \rangle_\alpha, \quad (40)$$

where

$$\mathbf{E}(\mathbf{k}_{\bar{\alpha}}) = \mathbf{I} + \langle \mathbf{D}(\mathbf{x}_\alpha) \mathbf{C}(\mathbf{k}_{\bar{\alpha}} \diamond \mathbf{x}_\alpha) \rangle_\alpha. \quad (41)$$

Note that for the invertibility it is necessary and sufficient to have $\det \mathbf{E}(\mathbf{k}_{\bar{\alpha}}) \neq 0$, $\forall \mathbf{k}_{\bar{\alpha}}$, see, e.g., [6]. It is obvious that $\mathbf{E}(\mathbf{k}_{\bar{\alpha}})$ and $\mathbf{E}^{-1}(\mathbf{k}_{\bar{\alpha}})$ are both staircase. ■

Lemma 2.3. *i) Suppose that $\mathcal{A} = \sum_{\alpha \in I_N} \mathcal{A}_\alpha$ is invertible. Then \mathcal{A}_\emptyset is invertible. ii) Suppose that $\mathcal{A} = \mathcal{I} + \sum_{\alpha \in J} \mathcal{A}_\alpha$ is invertible, where $J \subset I_N \setminus \emptyset$ and $\mathcal{A}_\alpha \in \mathcal{H}_\alpha$. Then $\mathcal{I} + \mathcal{A}_\alpha$ is invertible for each $\alpha \in J$ such that $|\alpha| = \min_{\beta \in J} |\beta|$.*

Proof. i) The proof repeats the arguments of the proof of Lemma 2.1. We will use the same notations as in Lemma 2.1 and 2.2. We see that if \mathcal{A}_\emptyset is non-invertible then by Lemma 2.2 the matrix $\mathbf{A}_\emptyset(\mathbf{k}^0)$ is non-invertible for some \mathbf{k}^0 lying strictly inside the cube of the volume h^N (recall that all functions are h -staircase). Hence there is $\mathbf{f} \in \mathbb{C}^M$ such that $\mathbf{A}_\emptyset(\mathbf{k}^0)\mathbf{f} = 0$. Then $\mathcal{A}\mathbf{f}\eta$ (η is defined in Lemma 2.1) can be arbitrary small (for small $\varepsilon > 0$, see Lemma 2.1) while $\|\mathbf{f}\eta\|_{L^2}$ is fixed. This means that \mathcal{A} is non-invertible (by Banach's Open Mapping Theorem). For ii) the proof is the same as for i) but for the zero vector we need to take the zero vector \mathbf{f} of $\mathbf{E}(\mathbf{k}_\alpha^0)$ multiplied by $\mathbf{C}(\mathbf{k})$ and η_α (see notations in Lemmas 2.1 and 2.2). We have that if $\mathcal{I} + \mathcal{A}_\alpha$ is non-invertible then $(\mathcal{I} + \mathcal{A}_\alpha)\mathbf{C}(\mathbf{k})\mathbf{f}\eta_\alpha$ can be arbitrary small (for small $\varepsilon > 0$, see Lemma 2.1) while the norm of $\mathbf{C}(\mathbf{k})\mathbf{f}\eta_\alpha$ is not small. The same property will be for $\mathcal{A}\mathbf{C}(\mathbf{k})\mathbf{f}\eta_\alpha$ since the norms of integrals $\langle \eta_\alpha \rangle_\beta$ are small for $|\beta| \geq |\alpha|$ and $\beta \neq \alpha$. All these arguments are the same as in [6], where they are presented with more details. ■

Lemma 2.4. *Suppose that $\prod_{\alpha \in I_N} \mathcal{G}_\alpha = \prod_{\alpha \in I_N} \tilde{\mathcal{G}}_\alpha$, where $\mathcal{G}_\alpha, \tilde{\mathcal{G}}_\alpha \in \mathcal{G}_\alpha$ and the orders of the terms in both products are the same. Then $\mathcal{G}_\alpha = \tilde{\mathcal{G}}_\alpha$ for all α .*

Proof. Denote $\mathcal{G}_\alpha = \mathcal{I} + \mathcal{A}_\alpha$, $\tilde{\mathcal{G}}_\alpha = \mathcal{I} + \tilde{\mathcal{A}}_\alpha$, where $\mathcal{A}_\alpha, \tilde{\mathcal{A}}_\alpha \in \mathcal{L}_\alpha$, $\alpha \neq \emptyset$. Expanding the products we obtain

$$\mathcal{G}_\emptyset + \sum_{\alpha \in I_N \setminus \emptyset} \mathcal{B}_\alpha = \tilde{\mathcal{G}}_\emptyset + \sum_{\alpha \in I_N \setminus \emptyset} \tilde{\mathcal{B}}_\alpha \quad (42)$$

with some $\mathcal{B}_\alpha, \tilde{\mathcal{B}}_\alpha \in \mathcal{L}_\alpha$. Using Lemma 2.1 we obtain that $\mathcal{G}_\emptyset = \tilde{\mathcal{G}}_\emptyset$. Moreover, all other terms are also equal $\mathcal{B}_\alpha = \tilde{\mathcal{B}}_\alpha$. We have

$$\mathcal{G}_\emptyset \mathcal{A}_{\{i\}} = \mathcal{B}_{\{i\}} = \tilde{\mathcal{B}}_{\{i\}} = \tilde{\mathcal{G}}_\emptyset \tilde{\mathcal{A}}_{\{i\}} \text{ or } \mathcal{A}_{\{i\}} \mathcal{G}_\emptyset = \mathcal{B}_{\{i\}} = \tilde{\mathcal{B}}_{\{i\}} = \tilde{\mathcal{A}}_{\{i\}} \tilde{\mathcal{G}}_\emptyset \quad (43)$$

depending on the order of terms in the product. Multiplying (43) by $\mathcal{G}_\emptyset^{-1} = \tilde{\mathcal{G}}_\emptyset^{-1}$ leads to $\mathcal{A}_{\{i\}} = \tilde{\mathcal{A}}_{\{i\}}$ and hence $\mathcal{G}_{\{i\}} = \tilde{\mathcal{G}}_{\{i\}}$ for all $i = 1, \dots, N$. The same arguments allow us to prove $\mathcal{G}_{\{i,j\}} = \tilde{\mathcal{G}}_{\{i,j\}}$ and so on. ■

Proof of Theorem 1.4. First identity in (10) follows from (6), second is simple. The statement that \mathcal{L}_α are algebras is also simple, and $\mathcal{L}_\alpha \cap \mathcal{L}_\beta = \{0\}$, $\mathcal{G}_\alpha \cap \mathcal{G}_\beta = \{\mathcal{I}\}$ for $\alpha \neq \beta$ follow from Lemma 2.1. The statement that \mathcal{G}_α is a group follows from Lemma 2.2. The closedness of \mathcal{G}_α and \mathcal{H}_α is obvious. Suppose that $\mathcal{A} = \sum_{\alpha \in I_N} \mathcal{A}_\alpha$ is invertible, where $\mathcal{A}_\alpha \in \mathcal{H}_\alpha$. By Lemma 2.3 \mathcal{A}_\emptyset is invertible. Then $\mathcal{A}_1 = \mathcal{A}_\emptyset^{-1} \mathcal{A} = \mathcal{I} + \sum_{\alpha \in I_N \setminus \emptyset} \mathcal{A}_{\alpha 1}$ is invertible, where $\mathcal{A}_{\alpha 1} = \mathcal{A}_\emptyset^{-1} \mathcal{A}_\alpha \in \mathcal{H}_\alpha$ (see (7)). By Lemma 2.3 $\mathcal{I} + \mathcal{A}_{\{1\}1}$ is invertible with $(\mathcal{I} + \mathcal{A}_{\{1\}1})^{-1} = \mathcal{I} + \mathcal{B}_{\{1\}} \in \mathcal{G}_{\{1\}}$ (note that $\mathcal{A}_{\{1\}1}$ can be zero). Then $(\mathcal{I} + \mathcal{B}_{\{1\}})\mathcal{A}_1 = \mathcal{I} + \sum_{\alpha \in I_N \setminus (\emptyset \cup \{1\})} \mathcal{A}_{\alpha 2}$ is invertible, where $\mathcal{A}_{\alpha 2} \in \mathcal{H}_\alpha$. Repeating these steps 2^N times we consistently remove the terms with indices \emptyset , $\{i\}$ ($i = 1, \dots, N$), $\{i, j\}$ ($i, j = 1, \dots, N$) and so on. Note that the order of the steps is important in the sense that the term with index α should be removed after the term with index β if $|\alpha| > |\beta|$. If $|\alpha| = |\beta|$ then the order of removing the terms is not important. Finally, we obtain that \mathcal{A}_{2^N} defined by

$$\mathcal{A}_{2^N} = \mathcal{I} + \mathcal{A}_{\{1, \dots, N\}2^N}, \quad \mathcal{A}_{\{1, \dots, N\}2^N} \in \mathcal{L}_{\{1, \dots, N\}} \quad (44)$$

is invertible with

$$(\mathcal{A}_{2^N})^{-1} = \mathcal{I} + \mathcal{B}_{\{1, \dots, N\}}, \quad \mathcal{B}_{\{1, \dots, N\}} \in \mathcal{L}_{\{1, \dots, N\}}. \quad (45)$$

Going through all the steps we obtain that

$$\mathcal{A}^{-1} = \prod_{\alpha \in I_N \setminus \emptyset} (\mathcal{I} + \mathcal{B}_\alpha) \mathcal{A}_\emptyset^{-1}, \quad \mathcal{A} = \mathcal{A}_\emptyset \prod_{\alpha \in I_N \setminus \emptyset} (\mathcal{I} + \mathcal{A}_{\alpha i_\alpha}), \quad (\mathcal{I} + \mathcal{A}_{\alpha i_\alpha})^{-1} = \mathcal{I} + \mathcal{B}_\alpha, \quad (46)$$

where the orders of terms in the products are the direct and inverse orders of the steps. The uniqueness follows from Lemma 2.4. ■

Lemma 2.5. *The mapping $\tau : \mathcal{L} \rightarrow \mathcal{C}$ is linear with $\|\tau\|_{\mathcal{L} \rightarrow \mathcal{C}} = M$ and $\tau(\mathcal{AB}) = \tau(\mathcal{BA})$.*

Proof. The linearity of τ is obvious. Due to the linearity and "multiplicativity" of the standard trace Tr of square matrices, (6), (7), (12), (2) and the facts that

$$\overline{\alpha} \setminus (\alpha \cup \beta) = \overline{\alpha \cup \beta}, \quad \overline{\alpha} \cap (\alpha \cup \beta) = \beta \setminus \alpha, \quad (\alpha \setminus \beta) \cup (\alpha \cap \beta) = \alpha \quad (47)$$

we obtain

$$\tau_{\alpha \cup \beta}(\langle \mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha \langle \mathbf{B}(\mathbf{k}, \mathbf{x}_\beta) \cdot \rangle_\beta) = \tau_{\alpha \cup \beta}(\langle (\mathbf{A} \circ \mathbf{B})(\mathbf{k}, \mathbf{x}_{\alpha \cup \beta}) \cdot \rangle_{\alpha \cup \beta}) \quad (48)$$

$$= \text{Tr} \int_{[0,1]^{|\alpha \cup \beta|}} \int_{[0,1]^{|\alpha \cap \beta|}} \mathbf{A}(\mathbf{k}_{\alpha \cup \beta} \diamond \mathbf{x}_{\alpha \cup \beta}, \mathbf{x}_{\alpha \setminus \beta} \diamond \mathbf{z}_{\alpha \cap \beta}) \mathbf{B}(\mathbf{k}_{\alpha \cup \beta} \diamond \mathbf{x}_{\beta \setminus \alpha} \diamond \mathbf{z}_{\alpha \cap \beta} \diamond \mathbf{x}_{\alpha \setminus \beta}, \mathbf{x}_\beta) d\mathbf{z}_{\alpha \cap \beta} d\mathbf{x}_{\alpha \cup \beta} \quad (49)$$

$$= \text{Tr} \int_{[0,1]^{|\alpha \cup \beta|}} \mathbf{A}(\mathbf{k}_{\alpha \cup \beta} \diamond \mathbf{x}_{\alpha \cup \beta}, \mathbf{x}_\alpha) \mathbf{B}(\mathbf{k}_{\alpha \cup \beta} \diamond \mathbf{x}_{\alpha \cup \beta}, \mathbf{x}_\beta) d\mathbf{x}_{\alpha \cup \beta} \quad (50)$$

$$= \text{Tr} \int_{[0,1]^{|\alpha \cup \beta|}} \mathbf{B}(\mathbf{k}_{\alpha \cup \beta} \diamond \mathbf{x}_{\alpha \cup \beta}, \mathbf{x}_\beta) \mathbf{A}(\mathbf{k}_{\alpha \cup \beta} \diamond \mathbf{x}_{\alpha \cup \beta}, \mathbf{x}_\alpha) d\mathbf{x}_{\alpha \cup \beta} \quad (51)$$

$$= \tau_{\alpha \cup \beta}(\langle \mathbf{B}(\mathbf{k}, \mathbf{x}_\beta) \cdot \rangle_\beta \langle \mathbf{A}(\mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha). \quad (52)$$

Due to the linearity of τ , (36) and (48)-(52) we obtain that $\tau(\mathcal{AB}) = \tau(\mathcal{BA})$. Definition of the norm 1.2 and $\|\tau(\mathcal{I})\|_{\mathcal{C}} = M$ immediately lead to $\|\tau\|_{\mathcal{L} \rightarrow \mathcal{C}} = M$. ■

Lemma 2.6. *The mapping π is multiplicative.*

Proof. Consider the function $\tilde{\pi} = (\tilde{\pi}_\alpha)$ defined by

$$\ln \tilde{\pi}_\alpha(\mathcal{I} + \mathcal{A}) = - \sum_{n=1}^{+\infty} \frac{(-1)^n \tau_\alpha(\mathcal{A}^n)}{n} \quad (53)$$

for $\mathcal{A} \in \mathcal{L}$ with $\|\mathcal{A}\|_{\mathcal{L}} < 1$. The inequalities (see also Theorem 1.3 and Lemma 2.5)

$$\|\tau(\mathcal{A}^n)\|_{\mathcal{C}} \leq M \|\mathcal{A}^n\|_{\mathcal{L}} \leq M \|\mathcal{A}\|_{\mathcal{L}}^n \quad (54)$$

show that the series (53) converges absolutely. Using (53), $\tau_\alpha(\mathcal{A}\mathcal{B}) = \tau_\alpha(\mathcal{B}\mathcal{A})$ (see Lemma 2.5), and repeating the arguments given for the standard traces and determinants, see, e.g., [8], we obtain

$$\ln \tilde{\pi}_\alpha(\mathcal{I} + \mathcal{A}) + \ln \tilde{\pi}_\alpha(\mathcal{I} + \mathcal{B}) = - \sum_{n=1}^{+\infty} \frac{(-1)^n \tau_\alpha(\mathcal{A}^n) + (-1)^n \tau_\alpha(\mathcal{B}^n)}{n} \quad (55)$$

$$= - \sum_{n=1}^{+\infty} \frac{(-1)^n \tau_\alpha((\mathcal{A} + \mathcal{B} + \mathcal{A}\mathcal{B})^n)}{n} = \ln \tilde{\pi}_\alpha((\mathcal{I} + \mathcal{A})(\mathcal{I} + \mathcal{B})) \quad (56)$$

if the norms of \mathcal{A} , \mathcal{B} , $\mathcal{A} + \mathcal{B} + \mathcal{A}\mathcal{B}$ less than 1. The properties (55), (56) show us that $\tilde{\pi}$ is multiplicative. Moreover, $\tilde{\pi}(\mathcal{I} + \mathcal{A}_\alpha) = \pi(\mathcal{I} + \mathcal{A}_\alpha)$ for small $\mathcal{A}_\alpha \in \mathcal{L}_\alpha$ because τ_α is the standard trace of finite-rank operators and π_α is the standard determinant of finite-rank operators. Now consider $\mathcal{I} + \lambda\mathcal{A}$ for $\lambda \in \mathbb{C}$. Going through the procedure from the proof of Theorem 1.4 (see (40), (41), (44)-(46)) we obtain that

$$\mathcal{I} + \lambda\mathcal{A} = \prod_{\alpha \in I_N} (\mathcal{I} + \langle \mathbf{A}_\alpha(\lambda, \mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha), \quad (57)$$

where the function \mathbf{A}_α depends on $\lambda \in \mathbb{C}$ as a rational function (here we use the fact that all functions are h -staircase in variables \mathbf{k}, \mathbf{x} and hence the integrals are just finite sums). Moreover $\mathbf{A}_\alpha(\lambda, \mathbf{k}, \mathbf{x}_\alpha) = 0$ for $\lambda = 0$ because $\mathcal{I} + \lambda\mathcal{A} = \mathcal{I}$ for $\lambda = 0$ and the representation it as the product is unique, see Lemma 2.4. Then for small $\lambda \in \mathbb{C}$ we have that

$$\tilde{\pi}(\mathcal{I} + \lambda\mathcal{A}) = (\tilde{\pi}_\alpha(\mathcal{I} + \langle \mathbf{A}_\alpha(\lambda, \mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha)) = (\pi_\alpha(\mathcal{I} + \langle \mathbf{A}_\alpha(\lambda, \mathbf{k}, \mathbf{x}_\alpha) \cdot \rangle_\alpha)) = \pi(\mathcal{I} + \lambda\mathcal{A}). \quad (58)$$

Due to the rational dependence on $\lambda \in \mathbb{C}$ we obtain that (58) is valid for all $\lambda \in \mathbb{C}$ for which $\|\lambda\mathcal{A}\|_{\mathcal{L}} < 1$. Let \mathcal{A}, \mathcal{B} be two invertible operators. Using the same arguments as above and the fact that $\tilde{\pi}$ is multiplicative we obtain that for small $\lambda \in \mathbb{C}$

$$\pi((\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))(\mathcal{I} + \lambda(\mathcal{B} - \mathcal{I}))) = \tilde{\pi}((\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))(\mathcal{I} + \lambda(\mathcal{B} - \mathcal{I}))) = \quad (59)$$

$$\tilde{\pi}(\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))\tilde{\pi}(\mathcal{I} + \lambda(\mathcal{B} - \mathcal{I})) = \pi(\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))\pi(\mathcal{I} + \lambda(\mathcal{B} - \mathcal{I})) \quad (60)$$

which is

$$\pi((\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))(\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))) = \pi(\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))\pi(\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I})) \quad (61)$$

for small $\lambda \in \mathbb{C}$. Due to the rational dependence on $\lambda \in \mathbb{C}$ we obtain that (61) is valid for all $\lambda \in \mathbb{C}$ for which $(\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))$ and $(\mathcal{I} + \lambda(\mathcal{A} - \mathcal{I}))$ are invertible, e.g. for $\lambda = 1$. Then $\pi(\mathcal{A}\mathcal{B}) = \pi(\mathcal{A})\pi(\mathcal{B})$. ■

Proof of Theorem 1.6. Many of the results follow from Lemmas 2.5 and 2.6. The identity $\pi(e^{\mathcal{A}}) = e^{\tau(\mathcal{A})}$ follows from (53) and $\pi = \tilde{\pi}$. The continuity of π in the vicinity of \mathcal{I} follows from the expansion (53). If $\mathcal{A} \in \text{Inv}(\mathcal{L})$ then the continuity in the vicinity of \mathcal{A} follows from the continuity in the vicinity of \mathcal{I} because the multiplicative property gives us

$$\pi(\mathcal{A} + \mathcal{B}) - \pi(\mathcal{A}) = \pi(\mathcal{A})(\pi(\mathcal{I} + \mathcal{A}^{-1}\mathcal{B})) - \pi(\mathcal{I}) \text{ and } \|\mathcal{A}^{-1}\mathcal{B}\|_{\mathcal{L}} \leq \|\mathcal{A}^{-1}\|_{\mathcal{L}}\|\mathcal{B}\|_{\mathcal{L}} \quad (62)$$

is small for small \mathcal{B} . ■

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